4次元多様体に埋め込まれた曲面の種数について

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Abstract

交叉形式

$^\forall \alpha \in H_2(X)$ $^\exists \Sigma \subset X$ s.t. $\alpha = [\Sigma]$

(1) Σの種数の評価

(2) その微分構造の区別への応用

surface := conn. closed oriented 2-mfd.

homology : over $\mathbb{Z}$

§1. Intersection form

Lem

X: simply connected closed 4-mfd

$\Rightarrow$ $H_1(X) = 0$, $H^1(X) = 0$

$H_3(X) = 0$, $H^3(X) = 0$.

$H_2(X)$ and $H^2(X)$ have no torsion.

Def

X: ori. conn. cpt C^∞ 4-mfd

$\Sigma \subset X$: smoothly embedded surface

$\alpha \in H_2(X)$ is represented by Σ, denoted by $\alpha = [\Sigma]$

$\Leftrightarrow$ $\iota_*([\Sigma]) = \alpha$

但し. $\iota : \Sigma \hookrightarrow X$

$[\Sigma] \in H_2(\Sigma)$

fundamental class. □

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Fact

X: ori. conn. cpt. C^∞ 4-mfd

$\Rightarrow \forall \alpha \in H_2(X)$ $\exists \Sigma \subset X$: smoothly embedded surf.

s.t. $\alpha = [\Sigma]$

Ref

松本幸夫 : 4次元のトポロジー

Gompf - Stipsicz : 4-manifolds and Kirby Calculus. □

Def

X: ori. conn. cpt C^∞ 4-mfd.

$$Q_X : H^2(X, \partial X) \times H^2(X, \partial X) \longrightarrow \mathbb{Z}$$
$$(\alpha, \beta) \longmapsto \langle \alpha \cup \beta, [X] \rangle$$

fundamental class

$[X] \in H_4(X, \partial X) \cong \mathbb{Z}$

is called the intersection form. □

Rem

(1) By Poincaré dual, $H^2(X, \partial X) \cong H_2(X)$

Q_X is defined on $H_2(X) \times H_2(X)$

$Q_X(\alpha, \beta) = \alpha \cdot \beta$ となる

(2). Q_X is symmetric bilinear form

i.e. (i) $Q_X(\alpha, \beta) = Q_X(\beta, \alpha)$

(ii) $Q_X(\alpha_1 + \alpha_2, \beta) = Q_X(\alpha_1, \beta) + Q_X(\alpha_2, \beta)$

(iii) $Q_X(\lambda \alpha, \beta) = \lambda \cdot Q_X(\alpha, \beta)$ $(\lambda \in \mathbb{Z})$

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(3) We view Q_X as matrix

$$Q_X = (a_i \cdot a_j)_{i,j} \qquad (a_1, \cdots, a_n \in H_2(X) : \text{basis})$$

(4) $Q_{\bar{X}} = -Q_X$ ($\bar{X}$: Xの逆向き). □

Fact

X: ori. conn. cpt C^∞ 4-mfd.

$\Sigma_1, \Sigma_2 \subset X$: smoothly embedded surface

$$Q_X([\Sigma_1],[\Sigma_2]) = \Sigma_1 \cdot \Sigma_2 := \sum_{p \in \Sigma_1 \cap \Sigma_2} \varepsilon(p)$$

$\varepsilon(p)$: 正の交わりなら $+1$, 負の 〃 -1.

Def

$Q : A \times A \to \mathbb{Z}$: s.b. form (symmetric bilinear)

A: 有限生成自由アーベル群

(1) $rk(Q) := rk A$

(2) $b_2^+(Q) :=$ Qの正の固有値の数

$b_2^-(Q) :=$ 〃 負の 〃

(3) Q is even $\Leftrightarrow$ $^\forall \alpha$. $Q(\alpha, \alpha) \equiv 0 \ (2)$.

Q is odd $\Leftrightarrow$ even でない

(4) Q is unimodular $\Leftrightarrow$ $\det Q = \pm 1$

(5) Q is definite $\Leftrightarrow$ $b_2^+(Q) = rk(Q)$ or $b_2^-(Q) = rk(Q)$

Q is indefinite $\Leftrightarrow$ definite でない. □

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Def

Q, Q' : s.b. form.

$Q \cong Q' \iff {}^{\exists}P \in GL(n;\mathbb{Z})$

s.t. ${}^tPQP = Q'$ (基底変換) □

Fact

X : simply conn. closed 4-mfd

$\Rightarrow Q_X$ is unimodular.

Fact

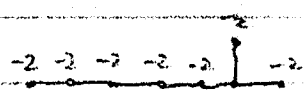
Q : indefinite unimodular s.b. form

(1) Q is odd $\Rightarrow Q \cong \begin{bmatrix} 1 & & & & & \\ & 1 & & & 0 & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & -1 & \\ & 0 & & & & \ddots \\ & & & & & & -1 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} b_2^+ \\ \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} b_2^- \end{matrix}$

(2) Q is even $\Rightarrow Q = \frac{|\sigma(Q)| - rk(Q)}{2} H \oplus \frac{\sigma(Q)}{8} E_8$.

where, $H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$E_8 = \begin{bmatrix} -2 & 1 & & & & & & \\ 1 & -2 & 1 & & & & 0 & \\ & 1 & -2 & 1 & & & & \\ & & 1 & -2 & 1 & & & 1 \\ & & & 1 & -2 & 1 & & 0 \\ & 0 & & & 1 & -2 & 1 & 0 \\ & & & & & 1 & -2 & 1 \\ & & & 1 & 0 & 0 & 1 & -2 \end{bmatrix}$$



$\rightsquigarrow$ Q は "odd or even" と b_2^+, b_2^- で決まる

$(\sigma(Q) = b_2^+ - b_2^-)$

signature

ex

(1) S^4. $H_2(S^4)=0$. $Q_{S^4}=\phi$.

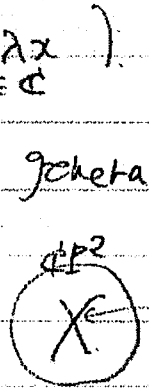
(2) $\mathbb{C}P^2$ $(:= \mathbb{C}^3\setminus\{0\}/x\sim\lambda x,\ \lambda\in\mathbb{C})$.

$H_2(\mathbb{C}P^2)=\mathbb{Z}$ generated by $\mathbb{C}P^1 \subset \mathbb{C}P^2$.

$\mathbb{C}P^1 = \{[z_1:z_2:z_1] \mid z_1=0\}$

$\qquad\qquad\qquad\qquad\qquad\qquad \{\quad\ \ \text{〃} \quad\ \ \mid z_2=0\}$

$Q_{\mathbb{C}P^2}=(+1)$



(3) $\overline{\mathbb{C}P^2}$: $\mathbb{C}P^2$の逆向き.

$Q_{\overline{\mathbb{C}P^2}}=(-1)$

(4). $S^2\times S^2$ $\quad H_2(S^2\times S^2)=\mathbb{Z}\oplus\mathbb{Z}$

generated by $[S^2\times pt]$, $[pt\times S^2]$

$$Q_{S^2\times S^2}=\begin{bmatrix}0&1\\1&0\end{bmatrix}. \leftarrow \text{even}.$$

(余談)

Thm [Akhmedov - Park, 先週]

exotic $S^2\times S^2$ exists.

↓

Thm [Rasmussen, 先週]

simp. conn. closed C^∞ 4-mfd で、1, 3 ハンドルなしのハンドル分解をもたないものが存在.

(5) $K3 = \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{C}P^3 \mid z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\}$

↳ cpx surface $\pi_1 = 0$.

$$H_2(K3) = \bigoplus_{22} \mathbb{Z}$$

$$Q_{K3} = 3H \oplus 2E_8.$$

(6) $X \# Y \rightsquigarrow Q_{X\#Y} = Q_X \oplus Q_Y$ $\square$.

Thm [Freedman]

$\forall Q$: unimodular s.b. form

$\exists$ simp. conn. closed C^0 4-mfd s.t. $Q_X \cong Q$

Furthermore

Q is even $\Rightarrow$ $\exists!$ such 4-mfd is unique

Q is odd $\Rightarrow$ $\exists$ exactly two such 4-mfds. at least one of which does not admit C^∞ str.

Thm. [Rochlin '52].

X: simp. conn. closed C^∞ 4-mfd.

Q_X: even $\Rightarrow$ $\sigma(Q_X) \equiv 0 \ (16)$.

(i.e. $Q_X = lH \oplus 2kE_8$, $l, k \in \mathbb{Z}$)

Thm [Donaldson '82].

X: simp. conn. closed C^∞ 4-mfd.

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Q_X: definite. $\Rightarrow$ $Q_X \cong \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$ or $\begin{bmatrix} -1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{bmatrix}$.

Cor

X: simp. conn. closed C^∞ 4-mfd

Q_X: odd $\Rightarrow$ $X \underset{\text{homeo}}{\approx} \#_n \mathbb{C}P^2 \#_m \overline{\mathbb{C}P^2}$.

Conj [Y. Matsumoto. 11/8-conjecture]

X: simp. conn. closed C^∞ 4-mfd.

$Q_X = \ell H \oplus 2k E_8 \Rightarrow \ell \geqq 3|k|$

$\left(\Leftrightarrow \frac{-K(Q_X)}{|\sigma(Q_X)|} \geqq \frac{11}{8} \right)$

正しければ $\xrightarrow{}$ Q_X : even $\Rightarrow X \underset{\text{homeo}}{\approx} \#_a K3 \#_b \overline{K3} \#_c S^2 \times S^2$.

Thm [Furuta]

10/8-inequality holds.

§2 種数の評価（古典的）

$\forall \alpha \in H_2(X)$, $\exists \Sigma \subset X$. $\alpha = [\Sigma]$

その最小種数は？

- 最小種数の上からの評価
  … 構成する（代数モノ, Kirby図式, 等々…）

- 下からの評価
  古典的な方法… intersection form の制限を用いる

Fact

X: ori. C^∞ 4-mfd

$\Sigma \subset X$: smoothly emb. sphere.

$\underset{\Sigma\cdot\Sigma}{\Sigma^2} = 1 \Rightarrow X \underset{\text{diff}}{\cong} {}^\exists X' \# \underset{\Sigma}{\mathbb{C}P^2}$.

Ex [Kervaire-Milnor '61]

$h \in H_2(\mathbb{C}P^2)$: 生成元

Then. $3h$ は. S^2 では表せない

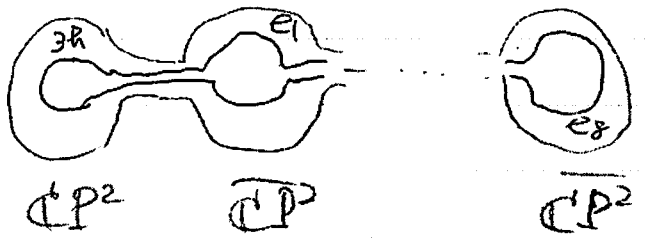
↑ torus では. 表せる

∵. $H_2(\overline{\mathbb{C}P^2}) \oplus \cdots \oplus H_2(\overline{\mathbb{C}P^2})$

$\quad e_1 \qquad\qquad e_8$: basis

$3h$ が S^2 で表せたとする

$\mathbb{C}P^2 \# 8\overline{\mathbb{C}P^2}$ で $3h+e_1+\cdots+e_8 =: \alpha$ は
S^2 で表せる.



ところで.

$\alpha^2 = 9-1-\cdots-1 = 1.$

∴ Fact より

$\exists$ simp. conn. closed C^∞ 4-mfd X s.t.

s.t. $\begin{cases} \mathbb{C}P^2 \# 8\overline{\mathbb{C}P^2} \underset{\text{diff}}{\cong} X \# \mathbb{C}P^2. \\ H_2(X) \oplus \underbrace{H_2(\mathbb{C}P^2)}_{\alpha} = H_2(\mathbb{C}P^2 \# 8\overline{\mathbb{C}P^2}) \end{cases}$

ところで.

$\alpha,\ \underbrace{h+e_1+e_2+e_3}_{\alpha_1},\ \underbrace{e_1-e_2}_{\alpha_2},\ e_2-e_3,\ \cdots,\ \underbrace{e_7-e_8}_{\alpha_8}$

は. $H_2(\mathbb{C}P^2 \# 8\overline{\mathbb{C}P^2})$ の基底

さらに $\alpha\cdot\alpha_i = 0$ $(1 \le i \le 8)$

∴ $\alpha_1, \cdots, \alpha_8 \in H_2(X)$: basis とみなせる

∴ $Q_X \cong E_8$

しかし. $\sigma(E_8) = -8 \neq 0$ (16)

∴ ロホリンの定理に矛盾. ▨

<u>Rem</u> Donaldson の定理を使うと. もっと色々わかる. □

§3. SW invariant と種数の評価

X : simp. conn. closed C^∞ 4-mfd with $b_2^+ > 1$ and odd

$SW_X : H^2(X) \longrightarrow \mathbb{Z}$: Seiberg-Witten invariant.

Fact

$$SW_{K3}(L) = \begin{cases} 1 & (L = 0) \\ 0 & (L \neq 0) \end{cases}$$

Thm [Taubes '94]

X : symplectic $\Rightarrow \exists K$ s.t. $SW_X(K) \neq 0$.

本来はもっと精密

Fact

$$SW_{(2n+1)\mathbb{CP}^2 \# m\overline{\mathbb{CP}^2}} \equiv 0$$

Thm (blow up formula)

$SW_X(K) \neq 0 \Rightarrow SW_{X \# \overline{\mathbb{CP}^2}}(K \pm E) \neq 0$.

($H^2(\overline{\mathbb{CP}^2})$ の生成元)

Thm (Adjunction inequality)

X : symplectic

$\Sigma \subset X$: smoothly emb. genus g surf. with $g \geq 1$.

$SW_X(K) \neq 0$.

$\Rightarrow |K([\Sigma])| + \Sigma \cdot \Sigma \leq 2g-2$.

Rem $g=0$ は使えない

§4 Exotic 4-mfd with ∂

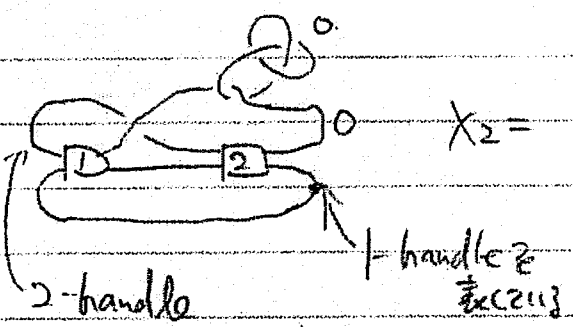
境界付き C^∞ 4-mfd の C^∞ str. の区別法.

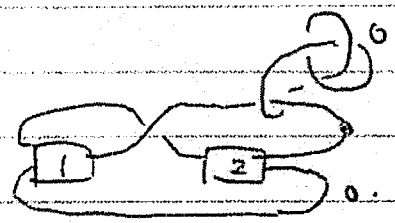
(1) SW inv. を計算 ⟶ ムズイ

(2) homology 類の種数を比べる

(3) 接着の仕方が一意な 4-mfd with ∂ をはりつけて closed 4-mfd を作る ⟶ その SW を計算.
↑ムズイ

(2) の例. [Akbulut-Y.]

$X_1 =$ 

$X_2 =$ 

[Lem $X_1 \underset{\text{homeo}}{\cong} X_2$]

☺ X_2 は X_1 から可縮な 4-mfd

を貼りなおしたもの ⟶ homeo. 図

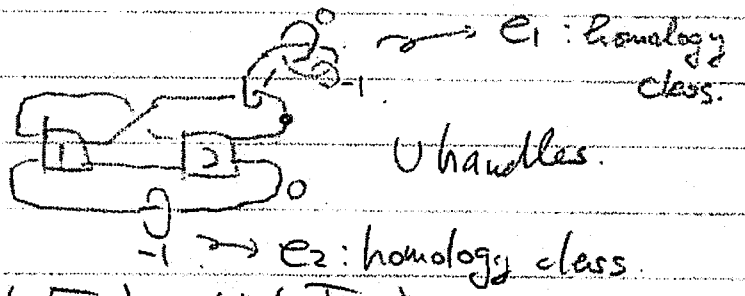
[Lem $H_2(X_1) \cong H_2(X_2) \cong \mathbb{Z}$]

$H_2(X_1)$ の生成元は torus で表せる.

∵ $\Sigma :=$ (trefoil の Seifert surface を D^4 の内部に押し込んだもの) $\cup$ (2-handle のコア $D^2 \times 0$)

が $H_2(X)$ を生成 ∎

Lem

$K3 \# 2\overline{\mathbb{CP}^2} =$ 

$$H_2(K3) \oplus \underset{\substack{\Psi \\ e_1}}{H_2(\overline{\mathbb{CP}^2})} \oplus \underset{\substack{\Psi \\ e_2}}{H_2(\overline{\mathbb{CP}^2})}$$

$\rightsquigarrow \alpha \in H_2(X_2)$: generator

$\rightsquigarrow \alpha \in H_2(X_2) \subset H_2(K3 \# 2\overline{\mathbb{CP}^2})$

$\rightsquigarrow \alpha \cdot e_i = 1 \quad (i = 1, 2)$

とおく.

Poincaré dual

$$SW_{K3 \# \overline{\mathbb{CP}^2}}(PD(e_1) + PD(e_2)) \neq 0$$

adjunction inequality より

$$\underbrace{PD(e_1+e_2) \cdot \alpha}_{=2} + \underbrace{\alpha \cdot \alpha}_{=0} \leq 2g-2.$$

$\therefore g \geq 2$. ∎