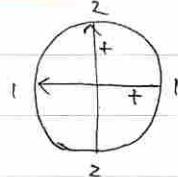
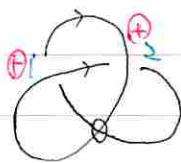


2 and 3-variations and finite type invariants of degree 2 and 3

$\text{レシ}_{\text{2}} \times 1^{\text{A}^0-\text{シ目}}$



{ All (long) virtual knots }  $\xleftrightarrow{1^{\text{A}^0}}$

{ All equivalence classes of  
Gauss diagrams by generalized  
Reidemeister moves }

$$\beta+1 \leftrightarrow \uparrow \leftrightarrow \beta-1$$

$$\begin{array}{c} \uparrow \\ \parallel \end{array} \leftrightarrow \begin{array}{c} \uparrow \\ \uparrow \end{array} \leftrightarrow \begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \end{array}$$

$$\begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \end{array} \leftrightarrow \begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \end{array} \leftrightarrow \begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \end{array}$$

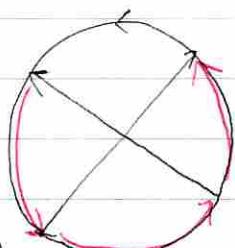
$\text{レシ}_{\text{2}} \times 2^{\text{A}^0-\text{シ目}}$

$$(\dots \rightarrow) = (\rightarrow) - (\quad)$$

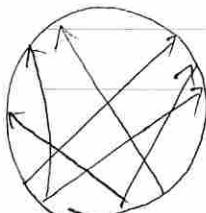
$$\begin{array}{c} \nearrow \\ \circ \\ \searrow \end{array} + \quad \begin{array}{c} \nearrow \\ \circ \\ \searrow \end{array} -$$

$\text{レシ}_{\text{2}} \times 3, 4^{\text{A}^0-\text{シ目}}$

$$\begin{array}{c} \uparrow \\ A_1 \varepsilon \\ \hline \uparrow \\ A_2 \end{array} \xrightarrow{\delta} \begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \end{array} \quad \begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \end{array} \xrightarrow{\quad} \begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \end{array}$$



(n-variation)



Thm.1 ([GPV])

$\mathcal{K} := \{\text{All (long) virtual knots}\}$

$U_n : \mathcal{K} \rightarrow \mathbb{Z}$  : an invariant of degree  $n$

- $n=1$

$U_1$  は constant map の  $\mathbb{Z}$

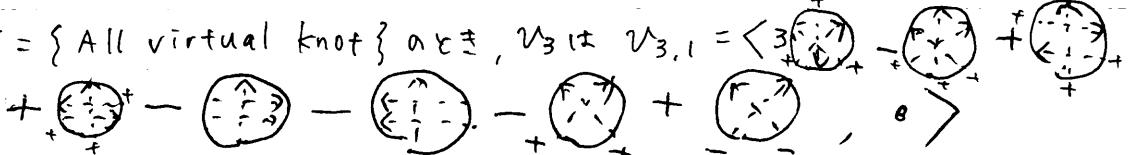
- $n=2$

$\mathcal{K} = \{\text{All virtual knots}\}$ ,  $U_2$  は constant map の  $\mathbb{Z}$

$\mathcal{K} = \{\text{All long virtual knots}\}$  かつ,  $U_2 \neq U_{2,1} = \left\langle \begin{array}{c} \text{Knot} \\ + + \end{array}, \bullet \right\rangle$ ,  
 $U_{2,2} = \left\langle \begin{array}{c} \text{Knot} \\ + + \end{array}, \bullet \right\rangle$  は  $\mathbb{Z}_2$  得られる。

- $n=3$

$\mathcal{K} = \{\text{All virtual knot}\}$  かつ,  $U_3 \neq U_{3,1} = \left\langle \begin{array}{c} \text{Knot} \\ + + - + - + + \end{array}, \bullet \right\rangle$



は  $\mathbb{Z}_2$  得られるが INV が唯一つだけ得られる

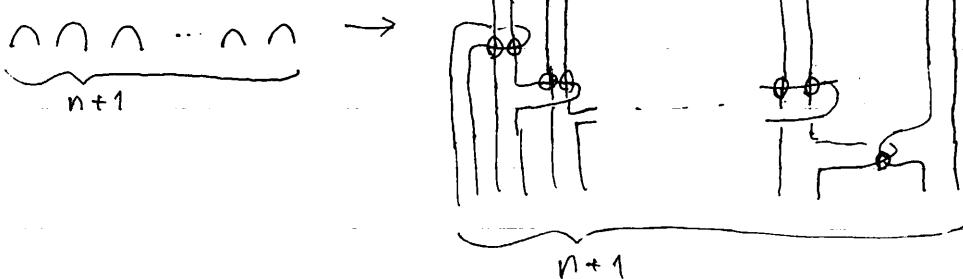
Example 2

- forbidden move (≠ 2-variation ([GPV]))

- $n=1$



- $n \geq 2$



Theorem 3

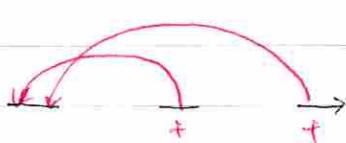
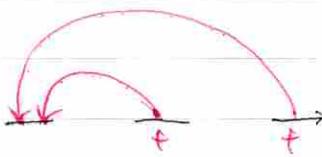
$D, D'$ : Gauss diagrams of long virtual knots s.t.  $D \xleftrightarrow{F} D'$

$$U_{2,1}(D) = U_{2,1}(D') = 0, \pm 1$$

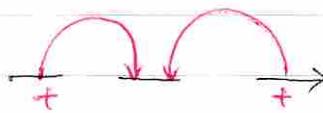
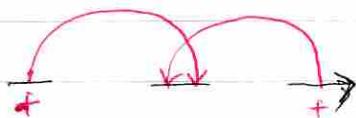
$$U_{2,2}(D) = U_{2,2}(D') = 0, \pm 1$$

Sketch proof

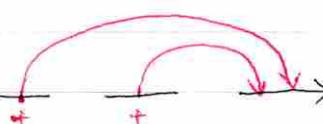
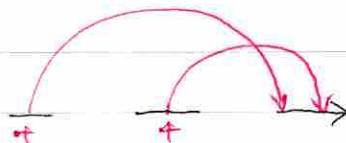
(1)

 $\xrightarrow{F}$ 

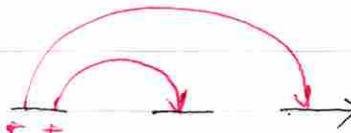
(2)



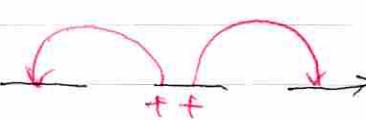
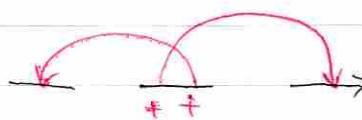
(3)



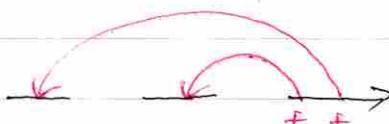
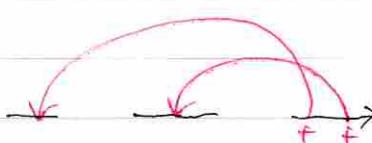
(4)



(5)



(6)



$\left\{ \begin{array}{l} \text{or } e^0 > 7 \text{ chordを持たない} \\ \text{or } e^0 > 7 \text{ chordを 1 本持つ } D \text{ の subdiagram} \end{array} \right\}$

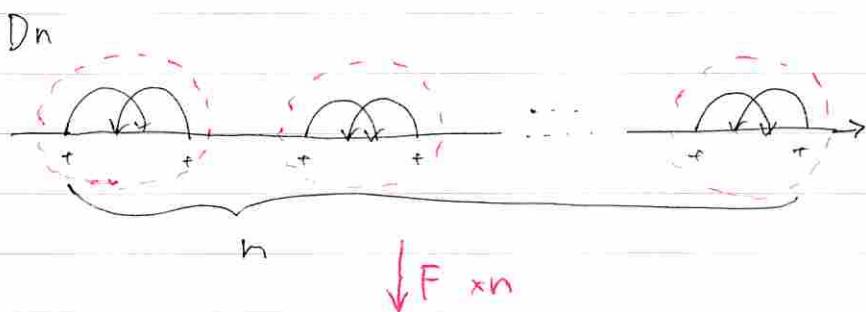
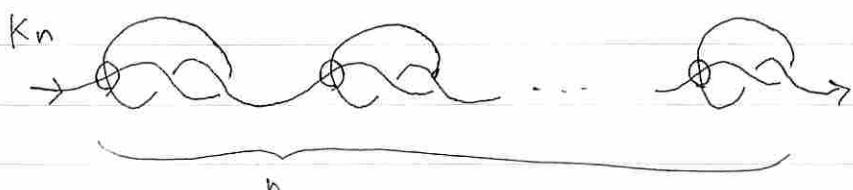
$= \left\{ \begin{array}{l} \text{or } e^0 > 7 \text{ chordを持たない 大阪市立大学理学部数学教室} \\ \text{or } e^0 > 7 \text{ chordを 1 本持つ } D' \text{ の subdiagram} \end{array} \right\}$

Case ①

$$\begin{aligned} \mathcal{U}_{2,1}(D) - \mathcal{U}_{2,2} &= \left\langle \begin{array}{c} \text{Diagram} \\ + \quad + \end{array} \right\rangle, \quad \left\langle \begin{array}{c} \text{Diagram} \\ + \quad + \end{array} \right\rangle, \quad \left\langle \begin{array}{c} \text{Diagram} \\ + \quad + \end{array} \right\rangle \\ &= \left( \begin{array}{c} \text{Diagram} \\ - \quad + \end{array}, \quad \begin{array}{c} \text{Diagram} \\ + \quad F \end{array}, \quad \begin{array}{c} \text{Diagram} \\ + \quad + \end{array} \right) \\ &= 0 \end{aligned}$$

② ~ ⑥ も 同様に 示す。 □

Example 4



$$\mathcal{U}_{2,1}(D_n) = \left\langle \begin{array}{c} \text{Diagram} \\ - \quad + \end{array}, \quad \begin{array}{c} \text{Diagram} \\ + \quad - \end{array}, \quad \dots, \quad \begin{array}{c} \text{Diagram} \\ - \quad + \end{array} \right\rangle = n$$

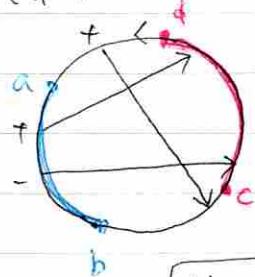
$$\mathcal{U}_F(K_n) = n$$

Notation

$D$ : the Gauss diagram of a trivial knot

$ab$ :  $D$  の circle 上の 2 点  $a, b$  を 向きに従ってつながる segment

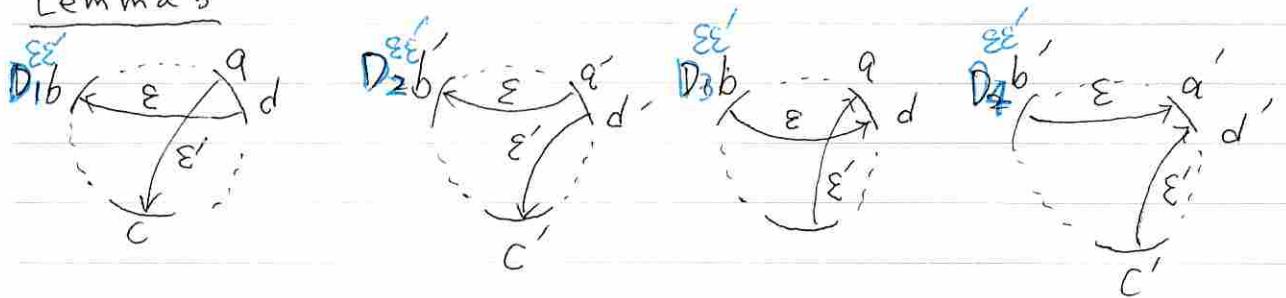
$cd$ :



$$N_{ab,cd} = 1 - 1 = 0$$

$N_{ab,cd} := (ab \vdash l = \text{tail}, cd \vdash l = \text{head} \in \mathbb{Z})$   
+ chord の本数)

$$- ( \quad )$$

Lemma 5

$$\begin{aligned} V_{3,1}(D_1^{\varepsilon\varepsilon'}) - V_{3,1}(D_2^{\varepsilon\varepsilon'}) &= \left\{ \begin{array}{l} -N_{ab,bc} + N_{bc,ab} + N_{bc,cd} - N_{cd,bc} - 1 \\ (\varepsilon = \varepsilon' = +1) \\ N_{ab,bc} - N_{bc,ab} - N_{bc,cd} + N_{cd,bc} \\ (\varepsilon \neq \varepsilon') \\ -N_{ab,bc} + N_{bc,ab} + N_{bc,cd} - N_{cd,bc} + 1 \\ (\varepsilon = \varepsilon' = -1) \end{array} \right. \end{aligned}$$

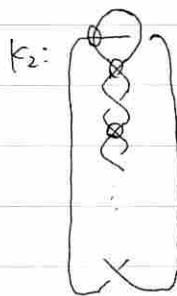
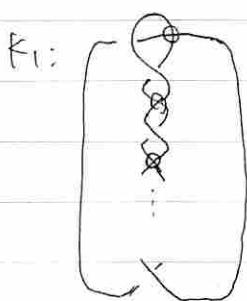
Theorem 6

$\forall n \in \mathbb{N}, \exists D_1, D_2 : \text{Gauss diagram}$ .

s.t.  $D \subsetneq D_2, V_{3,1}(D_1) - V_{3,1}(D_2) = n$

Proof

$D_1, D_2$ : 各  $\mathbb{R} K_1 \times K_2$  の Gauss diagram



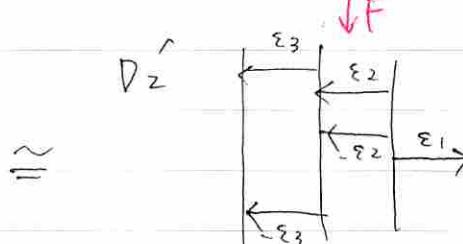
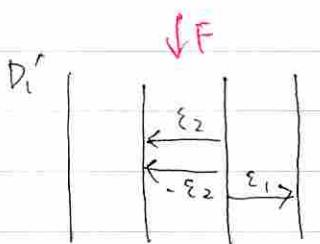
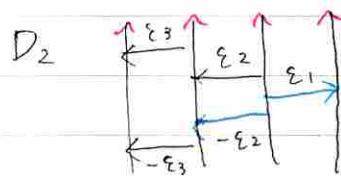
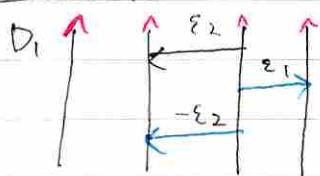
Lemma 5 より、

$$U_{3,1}(D_1) - U_{3,1}(D_2) = n$$

□

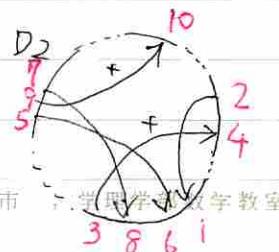
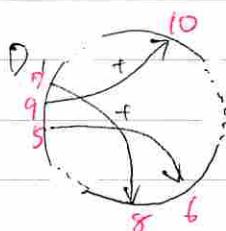
Thm 7

$D, D'$ : Gauss diagrams of virtual knots s.t.  $D \xrightarrow{(3)} D'$   
 $U_{3,1}(D) - U_{3,1}(D') = \pm 1$

Sketch proof

\* string の ORT の 2 通り 16通り

\* " " 2通り  $\times$  2通り  $= 4$ 通り



Lemma 5 ⑤).

$$v_{3,1}(D_1) = v_{3,1}(D'_1) - N_{98,8^{10}} + N_{8^{10},9^8} + N_{8^{10},10^7}$$

$$- N_{10^7,8^{10}} - 1$$

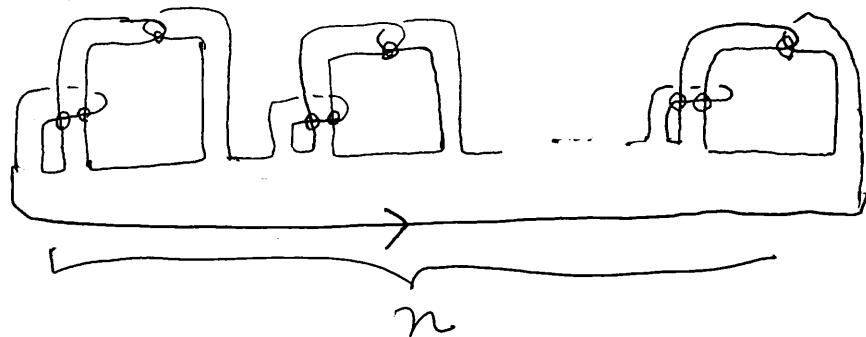
$$\left. \begin{aligned} v_{3,1}(D_2) &= v_{3,1}(D'_2) - (N_{98,8^{10}} + 1) + N_{8^{10},9^8} \\ &\quad + 1/N_{8^{10},10^7} - N_{10^7,8^{10}} - 1 \end{aligned} \right\}$$

$$v_{3,1}(D_1) - v_{3,1}(D_2) = 1$$

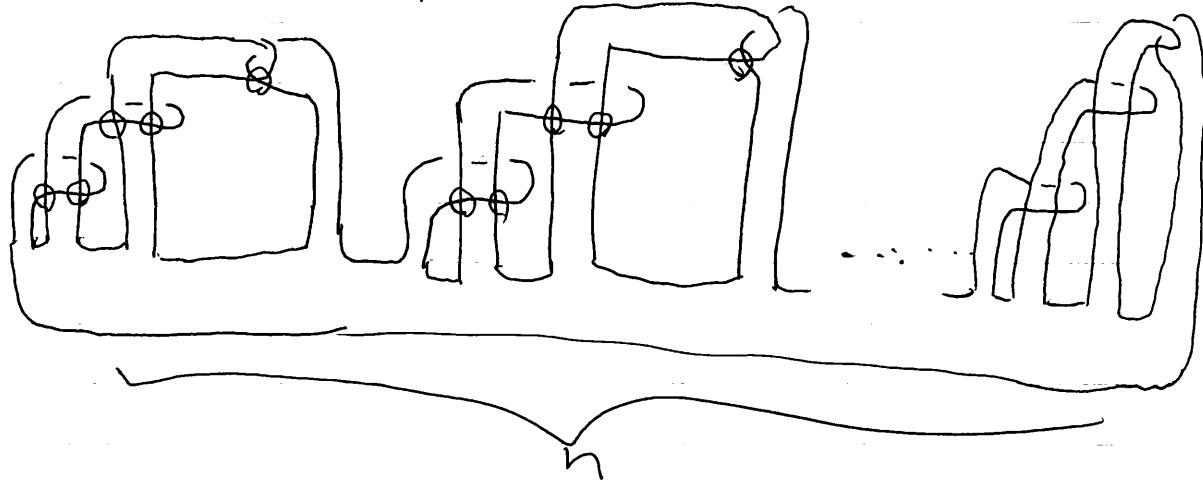
以下同様.  $\square$

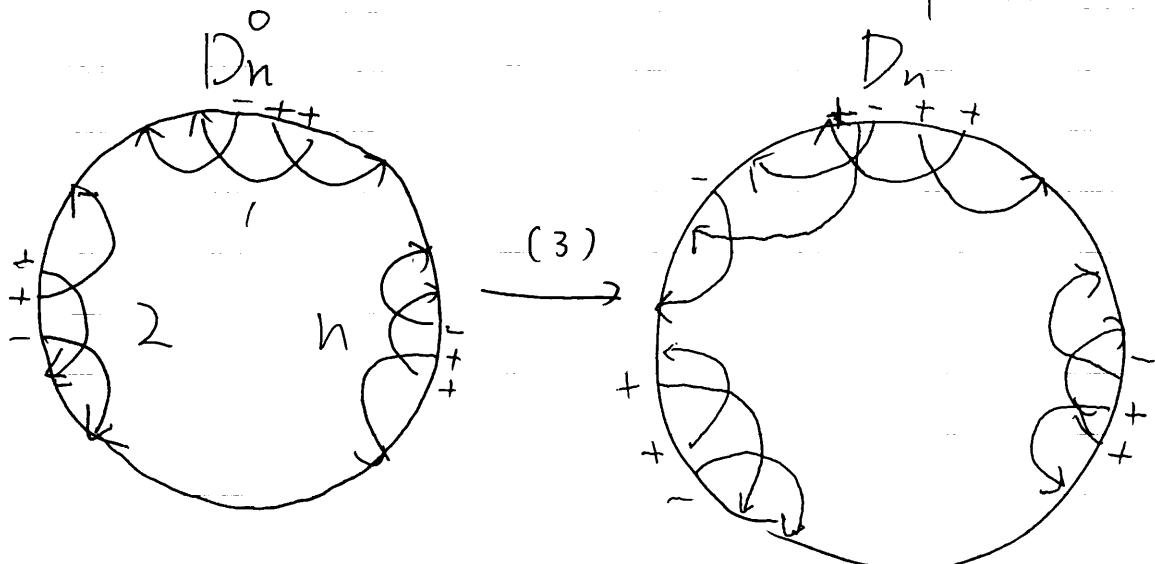
### Example 8

$K_n$ :



$K'_n$ :



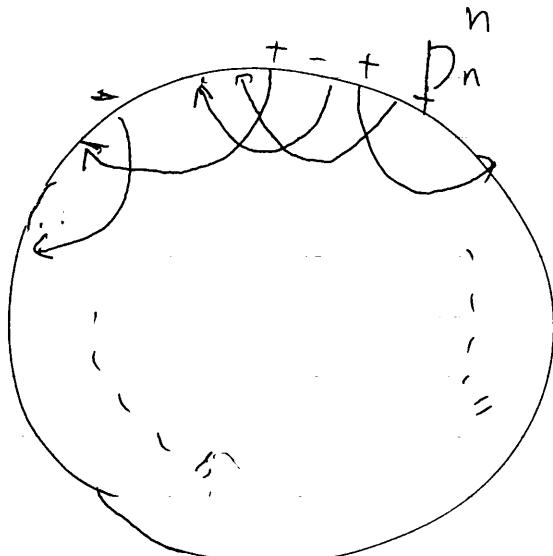


Thm 7 (3).

$$v_{3,1}(D_n^i) - v_{3,1}(D_n^{i+1}) = 1 \quad (i=0, 1, \dots, n-1)$$

$$\therefore v_{3,1}(D_n^0) - v_{3,1}(D_n^n) \underset{n}{\underbrace{\dots}}$$

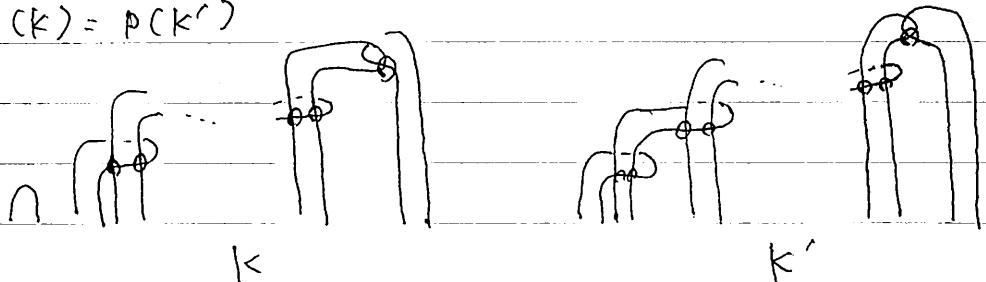
$$d_{(3)}(K_n, K_n') = n.$$



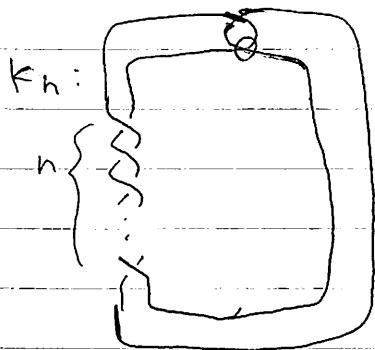
Thm 9

$K, K'$ : virtual knots s.t.  $K \xleftrightarrow{(n)} K' \quad (n \geq 3)$

$$P(K) = P(K')$$

Example 10

$K_n$  & trivial knot は 3-variation (3) の多角形をなす。



$$P(K_n) = \begin{cases} -n & (n: \text{even}) \\ n-1 & (n: \text{odd}) \end{cases}$$