

Chapter 1.

1.1 K-Theory

X : cpt Hausdorff space

$\text{Vect}(X) := \{ \text{the isom class of cpx vector bdl over } X \}$

$(\text{Vect}(X), \oplus)$: abelian semi-group.

Def

$$K(X) := \left\{ (E_0, E_1) \mid E_0, E_1 \in \text{Vect}(X) \right\} / \sim$$

where $(E_0, E_1) \sim (E'_0, E'_1)$

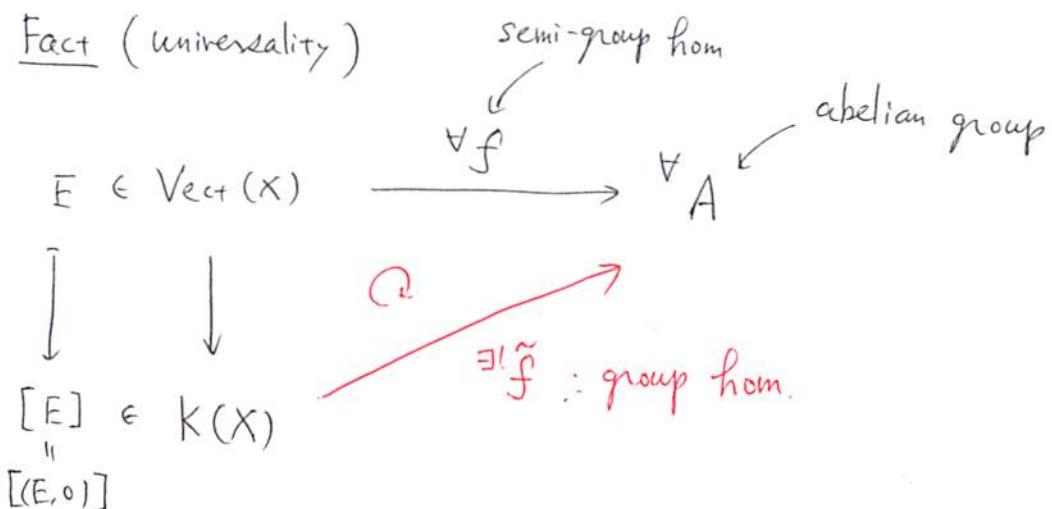
$$\stackrel{\text{def}}{\Leftrightarrow} \exists F \in \text{Vect}(X) \text{ s.t. } E_0 \oplus E'_1 \oplus F \cong E'_0 \oplus E_1 \oplus F$$

□

- (E_0, E_1) a class $\in [E_0] - [E_1] \in K(X)$ since

Rem $\oplus \approx \otimes \approx K(X)$ is abelian ring w/ identity.

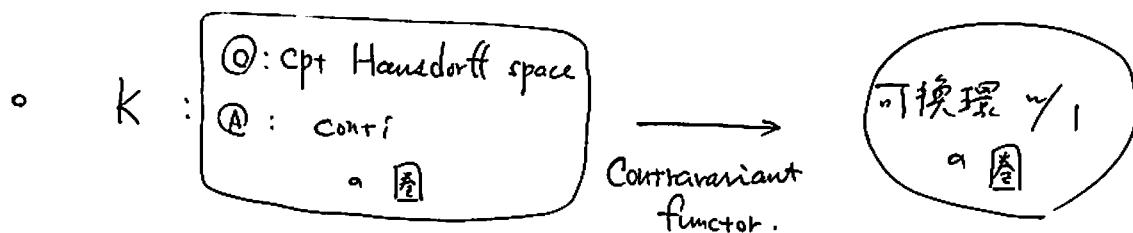
Fact (universality)



Property X, Y : cpt Hausdorff

$f: X \rightarrow Y$ conti \Rightarrow vector bundle or
pull-back π -induce.
 $\Rightarrow f^*: k(Y) \rightarrow k(X)$ ring hom

D.



(X, pt) : based cpt Hausdorff space.

$$pt \xhookrightarrow{i} X$$

$$\Rightarrow i^*: k(X) \rightarrow k(pt) \xrightarrow[\text{dim.}]{\cong \mathbb{Z}}$$

Def $\tilde{k}(X) := k \cap_{k(X)}^{i^* \text{ ideal.}} \underline{\text{the reduced } k\text{-group}}$

□

(X, Y) : X : cpt Hausdorff
 $Y \subset X$ closed

Def $k(X, Y) := \tilde{k}(X/Y) \xleftarrow{\text{base pt } (= Y/Y)}$

the relative k -group

D

Def X : loc cpt Hausdorff. X^+ : 1st cpt ft.

$$K_{cpt}(X) := \hat{K}(X^+) \quad \text{if } X \text{ is cpt.} - \not\in \text{base pt.} \quad \checkmark$$

the K -group with compact supports

□

Fact

$$K_{cpt}(X) = \left\{ (E_0, E_1 : \sigma_Y) \mid \begin{array}{l} Y \subset X \text{ cpt} \\ E_0|_{X-Y} \xrightarrow{\sigma_Y} E_1|_{X-Y} \text{ isom.} \end{array} \right\}$$

where $(E_0, E_1 : \sigma_Y) \sim (E'_0, E'_1 : \sigma_{Y'})$

$$\Leftrightarrow \left\{ \begin{array}{l} \exists (P, P : id_P), \exists (Q, Q : id_Q) \\ \text{s.t. } (E_0 \oplus P, E_1 \oplus P : \sigma_Y \oplus id_P) \cong (E'_0 \oplus Q, E'_1 \oplus Q : \sigma_{Y'} \oplus id_Q) \end{array} \right.$$

i.e. $\exists Z \supset Y \cup Y'$

$$\exists f_i : E_i \oplus P|_{X-Z} \xrightarrow{\cong} E'_i \oplus Q|_{X-Z}, i=0, 1$$

$$\text{s.t. } E_0 \oplus P|_{X-Z} \xrightarrow{\sigma_Y \oplus id_P} E_1 \oplus P|_{X-Z}$$

$$\begin{array}{ccc} f_0 \downarrow \cong & Q & \cong \downarrow f_1 \\ E'_0 \oplus Q|_{X-Z} & \xrightarrow{\sigma_{Y'} \oplus id_Q} & E'_1 \oplus Q|_{X-Z} \end{array}$$

Rem

$$z \in \mathbb{C}. \quad E_0 \xrightarrow{\sigma} E_1 \text{ from } z$$

$$\text{if } z \text{ cpt set } K \subset X \text{ s.t. } z \in K \subset X \quad E_0|_{X-K} \xrightarrow{\sigma|_{K^+}} E_1|_{X-K} \text{ isom.}$$

to object σ " ". $K_{cpt}(X)$ a π Σ 定義

□

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Theorem / Fact (Thom isom in K-Theory)

X : loc cpt Hausdorff

$E \xrightarrow{\pi} X$ cpx vector bundle w/ hermitian metric.

Then:

$$K_{\text{cpt}}(X) \xrightarrow{\cong} K_{\text{cpt}}(E) \quad \text{Thom isomorphism.}$$

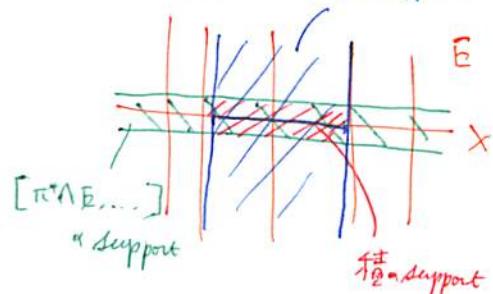
fr. 1.2 + 2.3 + 3.

$$\alpha \mapsto (\pi^* \alpha) \cdot [\pi^* \Lambda_{\text{ev}}^{\text{even}} E, \pi^* \Lambda_{\text{ev}}^{\text{odd}} E : \sigma]$$

$$\text{Total } \sigma_e(\varphi) = e \wedge \varphi - e^* \lrcorner \varphi. \quad \left\{ \begin{array}{l} \textcircled{1} \quad \text{well-def} \\ \text{at } \gamma \times \{ \} \\ K_{\text{cpt}}(E) \end{array} \right.$$

$$\textcircled{2} \quad \text{fiberwise "integration" = (index) } \approx 1$$

at $\gamma \times \{ \}$ $X \in \text{support}$



Theorem / Fact (Bott Periodicity)

$$K_{\text{cpt}}(X \times \mathbb{R}^2) \cong K_{\text{cpt}}(X)$$

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1.2. Elliptic differential Operator and Analytic index.

Def (differential operator on manifold)

X : manifold n -dim

$E, F \rightarrow X$: cpx vector bundle.

$P: P(E) \rightarrow P(F)$ linear map or differential operator of order m

\Leftarrow $\forall x \in X$, $\forall U$: local coord s.t. $U \cong \mathbb{R}^n$
 $(U, (x^1, \dots, x^n))$ $E|_U, F|_U$: triv.

$$\begin{aligned} \forall \text{ trivializations } E|_U &\xrightarrow{\sim} U \times \mathbb{C}^p \\ F|_U &\xrightarrow{\sim} U \times \mathbb{C}^q \end{aligned}$$

P は \mathbb{R}^n 形で表わす。

$$P = \sum_{|\alpha| \leq m} A^\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha} \quad \alpha = (\alpha', \dots, \alpha^n), \text{ multi index}$$

□

Symbol

最高次の微分は、簡単な変換を受ける。

loc. coord \rightarrow がりで

ex. $\varphi : (\mathbb{R}^n, (x_1, \dots, x_n)) \rightarrow (\mathbb{R}^n, (y_1, \dots, y_n))$ diffeo

$$\varphi(0) = 0$$

いたし。

$$\begin{aligned} \frac{\partial^2 f}{\partial y_\mu \partial y_\nu} &= \left(\sum_{k=1}^n \frac{\partial x_k}{\partial y_\mu} \frac{\partial}{\partial x_k} \right) \circ \left(\sum_{k=1}^n \frac{\partial x_k}{\partial y_\nu} \frac{\partial f}{\partial x_k} \right) \\ &= \underbrace{\sum_{\substack{j,k=1 \\ j,k}}^n \frac{\partial x_j}{\partial y_\mu} \frac{\partial x_k}{\partial y_\nu} \frac{\partial^2 f}{\partial x_j \partial x_k}} + \underbrace{\sum_{j,k=1}^n \frac{\partial^2 x_j}{\partial y_\mu \partial y_\nu} \frac{\partial f}{\partial x_k}} \end{aligned}$$

最高次の微分の part

- ・ 微分が一元化される。
- ・ その変換は Jacobian 乗

零次 part

- ・ 微分が変換函数
- ・ それが零次 part

$\rightarrow \{i^m A^\alpha\}_{|\alpha|=m}$ は $(\bigodot^m T X) \otimes \text{Hom}(E, F)$

a section $\sigma(P)$ を定めよ。

Definition

$\sigma(P) \in P$ の principal symbol は

- $\xi \in T_x^* X$ は $\sigma(P)$ の像

$$\sigma_\xi(P) : E_x \longrightarrow F_x$$

$\pi^* E \xrightarrow{\sigma(P)}$ bundle map

$$\begin{array}{ccc} & \downarrow & \\ \pi^* E & \xrightarrow{\sigma(P)} & \pi^* F \\ & \downarrow & \\ T^* X & \xrightarrow{\pi} & X \end{array}$$

$T_x X$ が元の contraction

linear map

$$\left\langle \frac{\partial}{\partial x_i}, i \sum \xi_i dx_i \right\rangle = i \xi_i$$

$$f' : \frac{\partial}{\partial x_i} \mapsto i \xi_i \text{ は } \text{相合}.$$

たゞ i は相合

→

Def (Elliptic Diff op)

P : diff op of order m on X が elliptic

$\Leftrightarrow \forall \xi \neq 0 \in T_x^* X \quad \sigma(P)$

$$\sigma_\xi(P) : E_x \longrightarrow F_x \quad \text{が isom.}$$

- X : cpt な \mathbb{R} ".

P : elliptic diff op

$$\rightarrow [\pi^* E, \pi^* F; \sigma(P)] \in k_{cpt}(T^* X)$$

K 群 が元が定まる

Facts

P : elliptic operator on closed mfd X

$\Rightarrow P$: Fredholm i.e. $\dim \ker P < \infty$, $\dim \text{Coker } P < \infty$

Def. (Analytic index)

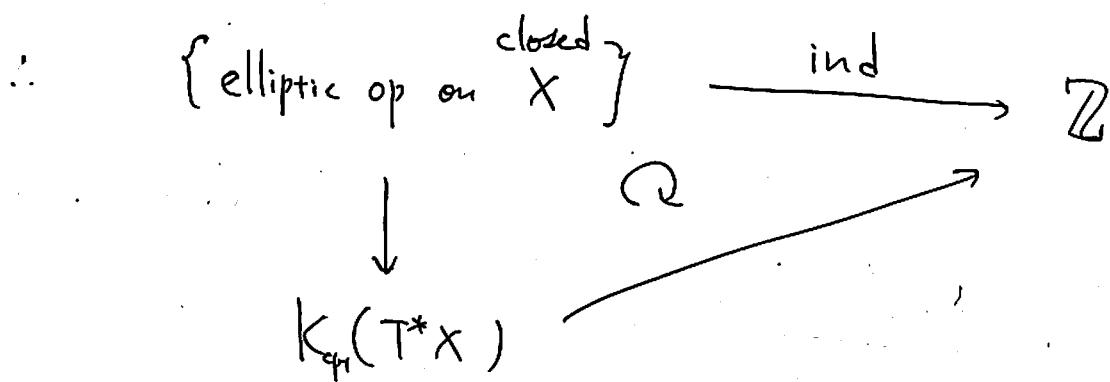
P : elliptic operator on closed mfd X a

(analytic) index $\in \mathbb{Z}$ def

$$\text{ind } P := \dim_{\mathbb{C}} \ker P - \dim_{\mathbb{C}} \text{Coker } P$$

Fact P, Q : elliptic operators on closed mfd X int'l

$$\sigma(P) = \sigma(Q) \Rightarrow \text{ind } P = \text{ind } Q.$$



(待問)

§ 1.3 Topological Index

Setting

X : closed C^∞ -mfld. $\dim = m$. w/ Riem metric fixed.

$E, F \rightarrow X$ C^∞ vector bundles

$P : \Gamma(E) \rightarrow \Gamma(F)$ elliptic operator.

principal symbol.

$$TX \xrightarrow{\pi} X$$

$$\sigma(P) := [\pi^* E, \pi^* F : \sigma(P)] \in K_{\text{cpt}}(TX)$$

if

$$K(DX, SX)$$

map $K_{\text{cpt}}(TX) \rightarrow \mathbb{Z}$

方針 $K_{\text{cpt}}(TX) \xrightarrow{\oplus} K_{\text{cpt}}(\mathbb{R}^N) \xrightarrow{\text{Bott Periodicity}} \mathbb{Z}$.

構成 $f : X \hookrightarrow \mathbb{R}^N$ smooth embedding (Whitney's embedding thm
 $(\exists f')$, 存在 f .)

$$\begin{array}{ccc} TX \hookrightarrow T\mathbb{R}^N & & N: \text{normal bundle} \\ \downarrow & \downarrow & \text{i.e. } N \\ X \hookrightarrow \mathbb{R}^N & & \downarrow X \\ & & \tilde{N} \\ & & \text{s.t. } TX \oplus N \cong T\mathbb{R}^N|_X \\ & & \downarrow \\ & & X = X \end{array}$$

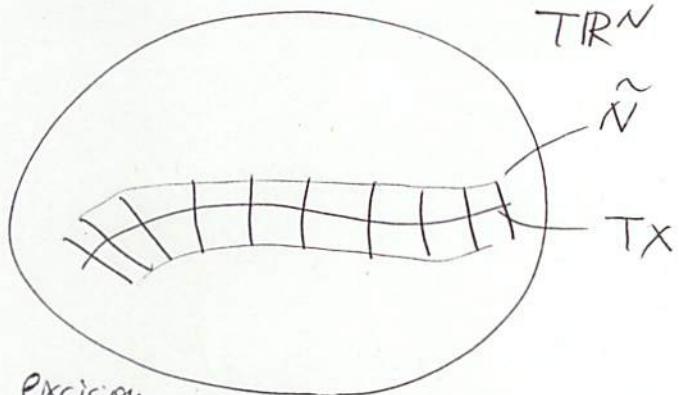
? TX $\hookrightarrow T\mathbb{R}^N$ a normal bundle?

$$\begin{array}{ccc} T(TX) \oplus \tilde{N} & \xrightarrow{N \oplus N} & \text{base fiber} \quad \text{fiber fib.} \\ \downarrow & \downarrow & \downarrow \\ TX \xrightarrow{\pi_X} X & & \tilde{N} \cong \pi_X^*(N \oplus N) \cong \pi_X^* N \otimes_{\mathbb{R}} \mathbb{C} \\ & & \text{real imaginary } \mathbb{R}, \mathbb{C}, \text{cpn strgs.} \end{array}$$

$N \rightarrow X$ is local $\Leftrightarrow (x, u)$ neighborhood

$$\begin{array}{ccc} TN \rightarrow TX \text{ is local} & \xrightarrow{TN \rightarrow TX \text{ a fiber}} & (x, \frac{\partial}{\partial x}) = TX \\ TN = (x, \frac{\partial}{\partial x}, \frac{\partial}{\partial u}) & \xrightarrow{\text{base fiber}} & (x, \frac{\partial}{\partial x}) \end{array}$$

tubenbd
 \downarrow
 $TX \subset \tilde{N} \subset TR^N$ ↣ 23.



$$K_{cpt}(TX) \xrightarrow[\text{isom.}]{} K_{cpt}(\tilde{N}) \xrightarrow{\text{excision}} K(TR^N) = K(R^{2n})$$

$f_!$

Definition (topological index)

elliptic operator P a $\overset{\text{principal}}{\underset{\text{symbol}}{\circ}}$ topological index ϵ .

$$\text{top-ind}(P) := f_! f_! \circ (P) \quad \text{def.}$$

Lemma

$\text{top-ind}(P)$ is independent of the choice of an embedding f

proof ($f_1 \in \mathcal{F}_n$ など)

(1)

① $j : \mathbb{R}^N \hookrightarrow \mathbb{R}^{N+N'} \in$ linear inclusion \approx 3c.

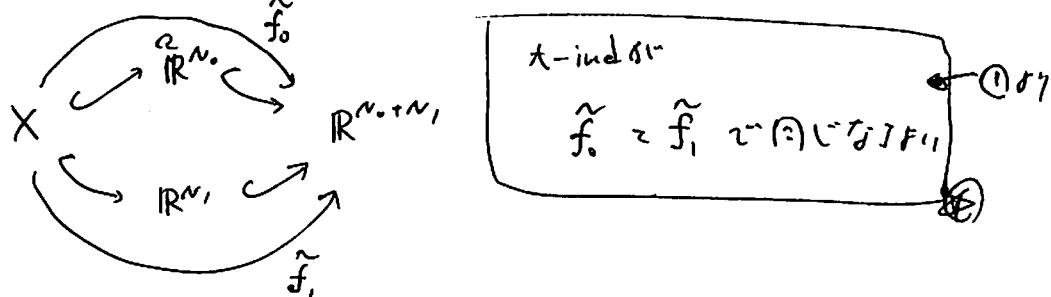
$j_! : K_{\text{cpt}}(\mathbb{T}\mathbb{R}^N) \longrightarrow K_{\text{cpt}}(\mathbb{T}\mathbb{R}^{N+N'})$ (Thom isom).

$\tilde{f} = j \circ f : X \hookrightarrow \mathbb{R}^{N+N'} \approx$ 3c.

$$\begin{array}{ccc}
 & K_{\text{cpt}}(TX) & \\
 f_! \searrow & \curvearrowleft & \tilde{f}_! \swarrow \\
 K(\mathbb{T}\mathbb{R}^N) & \xrightarrow[d_!]{\text{(Thom isom)}} & K(\mathbb{T}\mathbb{R}^{N+N'}) \\
 \swarrow \quad \curvearrowleft & & \swarrow \quad \curvearrowleft \\
 g_! \searrow & \mathbb{B} & \tilde{g}_! \swarrow \\
 (\text{Thom isom})^{-1} & & (\text{Thom isom})^{-1}
 \end{array}$$

$$f'_! \quad g_! \circ f_! = \tilde{g}_! \circ \tilde{f}_!$$

② $f_0 : X \hookrightarrow \mathbb{R}^{N_0}$, $f_1 : X \hookrightarrow \mathbb{R}^{N_1}$ \in 2c, 9 embedding \approx 3



iff. $\tilde{f}_0 \underset{\text{isotopic}}{\simeq} \tilde{f}_1$ by linear isotopy $\tilde{f}_t := (1-t)\tilde{f}_0 + t\tilde{f}_1$, $t \in [0, 1]$.
family of embeddings

K_{cpt} a homotopy invariance \Rightarrow $\tilde{f}_0_! = \tilde{f}_1_!$

$\therefore \text{④ } \text{Thom } f_1$

□

1.4. Atiyah-Singer Index Theorem

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Theorem (Atiyah-Singer)

X : closed mfd

P : elliptic operator on X

(= true. If $\sigma(P) \neq \emptyset$)

$$\text{index } P = \chi\text{-ind } P$$

$$(\overset{\text{"}}{\alpha\text{-ind}}(\sigma(P))) (\overset{\text{"}}{\lambda\text{-ind}}(\sigma(P)))$$

EII.
p.symbol. ↓

$$\begin{array}{ccc} & \text{ind.} & \\ Q. & \searrow & \\ K_{\text{top}}(TX) & \xrightarrow{\chi\text{-ind.}} & \mathbb{Z} \\ & \curvearrowright & \\ & (\alpha\text{-ind}) & \end{array}$$

$$\varepsilon_{ij} = \varepsilon$$

90%