

§ 3.3. Atiyah-Singer Index formula

(1)

Theorem (Atiyah-Singer)

X : closed mfd. (non-orientable $\Rightarrow \mathbb{Z}_2$) . $\dim X = n$.

P : elliptic differential operator on X

Then

$$\text{ind } P = (-1)^n \left\langle \text{ch}^{\circ}(P) \cdot \pi^* \text{Td}_{\mathbb{C}}(TX \otimes_{\mathbb{R}} \mathbb{C}), [TX] \right\rangle$$

proof $X \hookrightarrow \mathbb{R}^N$ embedding. $N \rightarrow X$ normal bundle.

diagram

$$\begin{array}{ccccccc}
 u \in K_{\text{cpt}}(TX) & \xrightarrow[\substack{\cong \\ \text{Thom}}]{i_!^k} & K(TN) & \xrightarrow[\substack{\cong \\ \text{extension}}]{f_*} & K(T\mathbb{R}^N) & \xleftarrow[\substack{\cong \\ \text{Thom}}]{i_!^k} & K(TP) = \mathbb{Z} \\
 \downarrow \text{ch}^{\circ} & \not\cong \textcircled{1} & \downarrow \text{ch}^1 & \textcircled{2} \quad \textcircled{3} & \downarrow \text{ch}^2 & \textcircled{3} & \downarrow \text{ch}^3 \\
 H_{\text{cpt}}^*(TX) & \xrightarrow[\substack{\cong \\ \text{Thom}}]{i_!^H} & H_{\text{cpt}}^*(TN) & \xrightarrow[\substack{\cong \\ \text{extension}}]{f_*} & H_{\text{cpt}}^*(T\mathbb{R}^N) & \xleftarrow[\substack{\cong \\ \text{Thom}}]{i_!^H} & H^*(TP) \cong \mathbb{Z} \\
 & & & & & & \downarrow \text{inclusion}
 \end{array}$$

$$t\text{-ind}(u) = i_!^{k-1} \circ f_* \circ i_!^k(u)$$

$TX \hookrightarrow T\mathbb{R}^N$ a normal bdl.

i.e. $\pi^* TN \otimes_{\mathbb{R}} \mathbb{C}$

$$= \text{ch}^3 \circ i_!^{k-1} \circ f_* \circ i_!^k(u)$$

$$\stackrel{\textcircled{3}}{=} i_!^{H-1} \left(\text{ch}^2 \circ f_* \circ i_!^k(u) \cdot (-1)^N \underbrace{\text{Td}_{\mathbb{C}}(\overline{T\mathbb{R}^N})^{-1}}_{\text{II}} \right)$$

$$\stackrel{\textcircled{2}}{=} (-1)^N i_!^{H-1} \left(f_* \circ \text{ch}^1 \circ i_!^k(u) \right)^{\text{II}}$$

$$= (-1)^N i_!^{H-1} \circ f_* \circ i_!^H \left((-1)^{N-n} \text{Td}_{\mathbb{C}}(\overline{TN})^{-1} \cdot \text{ch}(u) \right)$$

$$= (-1)^N i_!^{H-1} \circ f_* \circ i_!^H \left(\text{Td}_{\mathbb{C}}(\pi^* TX \otimes_{\mathbb{R}} \mathbb{C}) \cdot \text{ch}(u) \right)$$

- $TN \cong \pi^* N \otimes_{\mathbb{R}} \mathbb{C}$ & $TN \cong \overline{TN}$

$$\curvearrowright \text{Td}_{\mathbb{C}}(\overline{TN})^{-1} = \text{Td}_{\mathbb{C}}(TN)^{-1}$$

$$= \text{Td}_{\mathbb{C}}(T(TX))$$

- $TX \oplus N = T\mathbb{R}^N|_X$ & $T(TX) \oplus TN = T(T\mathbb{R}^N)|_{TX}$

$$= \text{Td}_{\mathbb{C}}(\pi^* TX \otimes_{\mathbb{R}} \mathbb{C})$$

$$\begin{aligned}
 \text{t-ind}(u) &= (-1)^n \frac{\tilde{i}^H}{2!^H} \circ \left(k \circ i^H(Td_{\mathbb{C}}(\pi^* TX \otimes_{\mathbb{R}} \mathbb{C}) \cdot ch(u)) \right) \\
 &\quad \xrightarrow{\text{fibrewise integration}} \left[\text{TR}^n \right] \\
 &= \left\{ (-1)^n k \circ i^H \circ \pi^* Td_{\mathbb{C}}(TX \otimes_{\mathbb{R}} \mathbb{C}) ch(u) \right\} \left[\text{TR}^n \right] \\
 &\quad \xleftarrow{\text{extension from } \mathbb{R} \text{ form } \mathbb{C}} \left[\text{TN} \right] \\
 &= \left\{ (-1)^n i^H \circ \pi^* Td_{\mathbb{C}}(TX \otimes_{\mathbb{R}} \mathbb{C}) ch(u) \right\} \left[\text{TN} \right] \\
 &\stackrel{?}{=} \left\{ (-1)^n \pi^* Td_{\mathbb{C}}(TX \otimes_{\mathbb{R}} \mathbb{C}) ch(u) \right\} \left[TX \right]
 \end{aligned}$$

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 2. 2. 3

\tilde{i}^H is Thom isom.
 \Rightarrow TX 上 \sim 積分才成立. i^H 之後 (Thom 矢量場)

TN 上 \sim 積分才成立 (同上)

$$u = \sigma(p) \in \text{主環} \text{ 得}$$

□

Theorem (Atiyah - Singer)

X : closed oriented n -dim mfd.

P : elliptic differential operator on X . $TX \xrightarrow{\pi} X$

Then,

$$\text{ind } P = (-1)^{\frac{n(n+1)}{2}} \left\{ \pi_! \text{ch} \sigma(P) \text{Td}_e(TX_{\mathbb{R}} \otimes \mathbb{C}) \right\} [X]$$

proof push-forward (fibrewise integration) $\pi_!$ 之前 Thm 1 用了

Q TX 上 ω 積分 = 各 fiber 上 ω 積分之和, X 上 ω 積分

∴ TX 上 ω 定義了

$$\begin{aligned} & (e_1, Je_1, \dots, e_n, Je_n) \\ & \downarrow \left((-1)^{\frac{n(n-1)}{2}} \right) \leftarrow \text{correction term} \\ & \underbrace{(e_1, \dots, e_n)}_{\text{base}}, \underbrace{(Je_1, \dots, Je_n)}_{\text{fiber}} \end{aligned}$$

推出了

↓ push-forward

$$(e_1, \dots, e_n)$$

$$\pi_! (\text{ch} \sigma(P), \pi^* \text{Td}_e(TX_{\mathbb{R}} \otimes \mathbb{C}))$$

$$\begin{aligned} \therefore \text{ind } P &= (-1)^{\frac{n(n-1)}{2}} (-1)^n \left\{ \pi_! \text{ch} \sigma(P) \cdot \text{Td}_e(TX_{\mathbb{R}} \otimes \mathbb{C}) \right\} [X] \\ &= (-1)^{\frac{n(n+1)}{2}} \left\{ \pi_! \text{ch} \sigma(P) \cdot \text{Td}_e(TX_{\mathbb{R}} \otimes \mathbb{C}) \right\} [X] \end{aligned}$$

□

3.4. Applications

[20]

Dirac Operator & principal symbol.

$$\begin{array}{ccc}
 \pi^* \mathcal{D}_C^+ & \xrightarrow{\sigma(\mathcal{D}^+)} & \pi^* \mathcal{D}_C^- \\
 \downarrow & \swarrow & \text{Diagram showing } \mathcal{D}^+ \text{ and } \mathcal{D}^- \text{ as curved arrows} \\
 T^*X & \xrightarrow{\pi} & X
 \end{array}$$

$$\xi = \sum_{i=1}^n \xi_i dx_i \in T_x^* X \text{ is null.}$$

$$\langle \mathcal{D}; \xi \rangle = \left\langle \sum_{i=1}^n e_i \cdot \frac{\partial}{\partial x_i}, i \sum_{i=1}^n \xi_i dx_i \right\rangle$$

$$= i \sum_{i=1}^n \xi_i e_i = i \xi.$$

$\xi \in TX \cong T_x^* X$ a Clifford action

$$\mathcal{D}(X) = \left[\pi^* \mathcal{D}_C^+, \pi^* \mathcal{D}_C^-; \underbrace{\sigma(\mathcal{D}^+)}_{\text{Clifford action}} \right] \in K_{\text{CPH}}(TX)$$

so \mathcal{D}^+ a principal symbol.

$\mathcal{D}^+ \in \mathcal{D}^+$: elliptic. σ p.s.

$$\underline{\pi_! \text{ch}(\mathcal{F}_C(X)) \text{ a } \text{算}} \quad TX \xrightarrow{\pi} X$$

$i: \text{O-section}$

$$\chi(X) \pi_! \text{ch}(\mathcal{F}_C(X)) = i^* \chi(\pi_! \text{ch}(\mathcal{F}_C(X)))$$

$$= i^* \text{ch}(\mathcal{F}_C(X))$$

$$= \text{ch}([\mathcal{F}_C^+], [\mathcal{F}_C^-])$$

splitting principle

$$\downarrow = \prod_{k=1}^n ([\bar{\ell}_k^{\frac{1}{2}}] - [\ell_k^{\frac{1}{2}}])$$

$$x_k = c_1(\ell_k)$$

$$= \prod_{k=1}^n \left(e^{-\frac{x_k}{2}} - e^{\frac{x_k}{2}} \right)$$

$$= (-1)^n \chi(X) \prod_{k=1}^n \frac{e^{\frac{x_k}{2}} - e^{-\frac{x_k}{2}}}{\frac{x_k}{2}}$$

$$= (-1)^n \chi(X) \hat{A}(X)^{-1}$$

$$\therefore \underline{\pi_! \text{ch}(\mathcal{F}_C(X)) = (-1)^n \hat{A}(X)^{-1}.}$$

Theorem (Atiyah-Singer)

X : closed spin mfd of $\dim = 2n$.

Consider the Dirac Operator

$$\mathcal{D}^+ : \Gamma(\mathcal{S}_e^+(X)) \longrightarrow \Gamma(\mathcal{S}_e^-(X))$$

Then

$$\text{ind } \mathcal{D}^+ = A(X)$$

proof Atiyah-Singer's index formula $\delta\gamma$

$$\text{ind } \mathcal{D}^+ = (-1)^{\frac{n(n+1)}{2}} \left\{ \text{Tr} \text{cho}(p) \cdot Td_e(TX_{\mathbb{R}} \otimes \epsilon) \right\} [X]$$

$$= \underbrace{(-1)^{\frac{2n(n+1)}{2}}}_{n} \left\{ (-1)^n \hat{A}(X)^T \hat{A}(X)^* \right\} [X]$$

$$(-1)^{2n^2+n+n} = 1$$

$$= \hat{A}(X)$$

□.

Theorem (Rochlin)

X : (smooth) closed spin mfd (=full).

$$\text{sgn}(X) \neq 16^{\frac{1}{2}} + 1 + 3.$$

proof X : 4-mfd (=full).

$$\mathbb{L}(X) = \frac{1}{3} p_1(X) [X]$$

$$\hat{A}(X) = -\frac{1}{24} p_1(X) [X]$$

$$\therefore \mathbb{L}(X) = -8 \cdot \hat{A}(X)$$

Hirzebruch signature thm 3.3 $\text{sgn}(X) = \mathbb{L}(X)$

Atiyah-Singer index thm 3.3 $\hat{A}(X) = \text{ind}(D^\circ) \in \mathbb{Z}$

$$\therefore \mathbb{L}(X) \equiv 3 \pmod{2^{\frac{1}{2}} + 1 + 3 \equiv 1 \pmod{3}}$$

$\hat{A}(X) \equiv 3 \pmod{2^{\frac{1}{2}} + 1 + 3 \equiv 1 \pmod{3}}$

$$\hat{A}(X) = \text{ind}(D^\circ) = \dim_{\mathbb{C}} \ker D^\circ - \underbrace{\dim_{\mathbb{C}} \text{Coker } D^\circ}_{\dim_{\mathbb{C}} \ker D^\dagger} (\because (D^\circ)^\dagger = D^\dagger)$$

$$\cong \mathbb{H}^2. \quad Cl_4 = \mathbb{H}(2) \cap \overset{\text{irr. rep}}{\mathbb{H}^2} \overset{\mathbb{H}^+ \oplus \mathbb{H}^-}{\cong} \underset{i,j}{\mathbb{Q}}$$

$$\therefore Cl_4 = \mathbb{C}(4) \cap \overset{\text{irr. rep}}{\mathbb{C}^4} = \mathbb{H}^2 = \overset{\text{even}}{\mathbb{C}} \oplus \overset{\text{odd}}{\mathbb{C}} = \mathbb{H}^+ \oplus \mathbb{H}^-$$

Clifford action on $\mathbb{H}(2) \cap \mathbb{H}^2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$,

Dirac operator $D(D^\dagger, D^\circ)$ i.e. i, j 作用 $\cong -T$ 構造

$\therefore \ker D^\circ, \text{Coker } D^\circ$ は (\mathbb{C} -vec sp の $\mathbb{Z}_2 \times \mathbb{Z}_2$) \mathbb{H}^1 -vec sp の構造をも?

$\therefore \dim_{\mathbb{C}} \cong \mathbb{Z}_2 \otimes \mathbb{Z}_2$. even.

□