§. Milnor's $\bar{\mu}$-invariants of links

L : an ordered ori. n-comp. link in S^3

$G := \pi_1(S^3 \setminus L)$

G_q : the qth lowercentral subgroup of G, ie, $G_1 = G$, $G_q = [G, G_{q-1}]$

$(G_q \triangleleft G,\ G_{q+1} \triangleleft G_q)$

Thm (Chen '52, Milnor '57)

the ith meridian

the ith longitude

$$G/G_q \cong \langle \alpha_1, \cdots, \alpha_n \mid [\alpha_i, \lambda_i]\ (i = 1, \cdots, n),\ A_q \rangle. \quad (A = \langle \alpha_1, \cdots, \alpha_n \rangle)$$

$\rightsquigarrow [\lambda_j] \in G/G_q$ is represented by a word of $\alpha_1, \cdots, \alpha_n$.

$\quad \| $

$\quad \lambda_j^q$ とかく

次に λ_j^q の Magnus 展開 E を考える. 整数係数 非可換無限べキ級数環

homo. $E : \langle \alpha_1, \cdots, \alpha_n \rangle \longrightarrow \mathbb{Z}\langle\langle X_1, \cdots, X_n \rangle\rangle$ is defined by

$$\begin{cases} E(\alpha_i) = 1 + X_i \\ E(\alpha_i^{-1}) = 1 - X_i + X_i^2 - X_i^3 + \cdots \end{cases}$$ ← $\frac{1}{1+X_i}$ の Maclaurin 展開

Prop. (Magnus - Karras - Solitar '66)

E is an injection. ie, $\langle \alpha_1, \cdots, \alpha_n \rangle \cong E(\langle \alpha_1, \cdots, \alpha_n \rangle)$

$$E(\lambda_j^q) = 1 + \sum_{k<q} \underbrace{\mu^{(q)}(i_1 \cdots i_k j)}_{X_{i_1} \cdots X_{i_k} \text{ の係数}} X_{i_1} \cdots X_{i_k} + (\text{terms of degree} \geq q)$$

(ただし、$\mu^{(q)}(j) = 0$ と定義する)

Rmk. $\underline{\mu^{(q)}(i_1 \cdots i_k j)}$ は、λ_j^q の表示に依存し不変量ではない.

Milnor number と呼ばれる

Prop. $w \in A_q$, $E(w) = 1 + (\deg \geq q)$

$\lambda_j^q \in G/G_q$, $\lambda_j^{q'} \in G/G_{q'}$ $(q < q')$ に対し、$\underline{\lambda_j^q \equiv \lambda_j^{q'} \pmod{A_q}}$ が

p. 8 の Claim (2q) より

成り立つので Prop より

$$E(\lambda_j^q) - E(\lambda_j^{q'}) = (\deg \geq q)$$

$$\therefore \mu^{(q)}(i_1 \cdots i_k j) = \mu^{(q')}(i_1 \cdots i_k j)$$

Note 以後 q を十分大きくとるものとして $\mu^{(q)}$ を μ と表す.

<u>Prop.</u> (Magnus – Karass – Solitar '66)

E is an injection , ie, $E(\langle \alpha_1, \cdots, \alpha_n \rangle) \cong \langle \alpha_1, \cdots, \alpha_n \rangle$

<u>Proof</u> $w = \alpha_{i_1}^{\varepsilon_1 n_1} \times \cdots \times \alpha_{i_r}^{\varepsilon_r n_r}$: reduced word ($n_i \in \mathbb{N}$, $\varepsilon_i \in \{-1, 1\}$, $\alpha_{i_m} \neq \alpha_{i_{m+1}}$)

$E(\alpha_{i_m}^{\pm 1}) = 1 + X_{i_m}$ or $1 - X_{i_m} + X_{i_m}^2 - X_{i_m}^3 + \cdots$ より

$$E(\alpha_{i_m}^{\varepsilon_m n_m}) = 1 + \varepsilon_m n_m X_{i_m} + X_{i_m}^2 \underbrace{f_{i_m}}_{\text{多項式}}$$ とかける.

$\therefore E(w) = (1 + \varepsilon_1 n_1 X_{i_1} + X_{i_1}^2 f_{i_1}) \times \cdots \times (1 + \varepsilon_r n_r X_{i_r} + X_{i_r}^2 f_{i_r})$

$X_{i_m} \neq X_{i_{m+1}}$ より $E(w)$ における $X_{i_1} \times \cdots \times X_{i_r}$ の係数は

$$\varepsilon_1 \times \cdots \times \varepsilon_r \times n_1 \times \cdots \times n_r$$

ここで $w \neq 1 \Rightarrow \varepsilon_1 \cdots \varepsilon_r n_1 \cdots n_r \neq 0$

$\Rightarrow E(w) \neq 1$

$\therefore \operatorname{Ker} E = \{1\}$ //

<u>Prop.</u> $w \in A_g \Rightarrow E(w) = 1 + $ (terms of degree $\geq g$)

<u>Proof.</u> induction on g

$g = 1 \Rightarrow A_1 = A \quad \therefore$ o.k.

$g > 1$ のとき (w が生成元の場合のみ考えればよい)

$w = [u, v] \quad (u \in A, v \in A_{g-1})$,

$$\begin{cases} E(u) = 1 + f, \ E(u^{-1}) = 1 + \bar{f} & (\deg f, \deg \bar{f} \geq 1) \\ E(v) = 1 + g, \ E(v^{-1}) = 1 + \bar{g} & (\deg g, \deg \bar{g} \geq g - 1) \end{cases}$$

$$\begin{aligned} E(w) &= E(u^{-1} v^{-1} u v) \\ &= (1 + \bar{f})(1 + \bar{g})(1 + f)(1 + g) \\ &= 1 + g + (1 + \bar{f})\bar{g}(1 + f) \\ &= 1 + g + \bar{g}(1 + f) + \bar{f}\bar{g}(1 + f) \\ &= 1 + g + \bar{g} + (\underbrace{\bar{g}f}_{\geq g-1 \ \geq 1} + \bar{f}\bar{g} + \bar{f}\bar{g}f) \\ &= 1 + g + \bar{g} + (\deg \geq g) \\ &= 1 - g\bar{g} + (\deg \geq g) = 1 + (\deg \geq g) \ // \end{aligned}$$

$$\boxed{\begin{aligned} \text{注)} \ & (1 + g)(1 + \bar{g}) = 1 \\ & = 1 + g + \bar{g} + g\bar{g} \\ \therefore \ & g + \bar{g} = -g\bar{g} \end{aligned}}$$

Def. (Milnor's $\bar{\mu}$-invariants)

$\#$ of comp. of L

$I = i_1 \cdots i_k j$: a sequence ($k < q$, $1 \le i_1, \cdots, i_k, j \le n$) $\in G/G_q$

$\mu(I)$: the Milnor number ie, the coeff. of $X_{i_1} \cdots X_{i_k}$ in $E(\lambda_j^{(q)})$

$$\Delta_L(I) := \gcd\left\{ \mu(J) \,\middle|\, J : \begin{array}{l} \text{Iから1つ以上の項を取り除き} \\ \text{残りを巡回置換して得られる数列} \end{array} \right\}$$

$$\bar{\mu}_L(I) := \mu(I) \mod \Delta_L(I)$$ ← これ自体 L の inv.

Milnor's $\bar{\mu}$-inv. of L

($\Delta_L(I) = 0 \Rightarrow \bar{\mu}_L(I) = \mu(I) \in \mathbb{Z}$)

Thm (Milnor '57)

$\bar{\mu}_L(I)$ is an inv. of L for any seq. I

Rmk $L = K_1 \cup \cdots \cup K_n$: n-comp. link.

- $\bar{\mu}_L(ij) = lk(K_i, K_j)$
- $\bar{\mu}_L$ is $\begin{cases} \text{an isotopy inv. (Milnor '57)} \rightsquigarrow \bar{\mu}_K = 0 \quad (K : \text{knot}) \\ \text{a link-concordance inv. (Casson '75).} \end{cases}$
- For a non-repeated seq I, $\bar{\mu}_L(I)$ is a link-homotopy inv. (Milnor '54, '57)

Rmk

Thm (Cochran '90) L, L' : links in S^3

$\bar{\mu}_L(I) = \bar{\mu}_{L'}(I) = 0$ for any seq. I with $|I| \le k$

$\Rightarrow \bar{\mu}_{L \#_b L'}(J) = \bar{\mu}_L(J) + \bar{\mu}_{L'}(J)$ for $\forall J$: seq. with $|J| \le k+1$

band sum

Thm (Krushkal '98) L, L' : links in S^3

$\forall I$: seq, $\Delta_{L \#_b L'}(I)$ is a multiple of $\gcd\{\Delta_L(I), \Delta_{L'}(I)\}$,

and

$$\bar{\mu}_{L \#_b L'}(I) \equiv \bar{\mu}_L(I) + \bar{\mu}_{L'}(I) \mod \gcd\{\Delta_L(I), \Delta_{L'}(I)\}.$$