Thm (Habiro '00)

$$L \overset{C_k}{\sim} L' \iff {}^{\exists} T_1 \cup \cdots \cup T_m = \text{a disj. union of simple } C_k\text{-trees for } L \text{ s.t. } L' \overset{a.i}{\sim} L_{\bigcup_{i=1}^{m} T_i}$$

Cor

${}^{\forall} L$: link, ${}^{\exists} G$: disj. union of C_1-trees s.t. $(\text{trivial link})_G \overset{a.i.}{\sim} L$.

Def. T : a simple C_k-tree for L

$$T = \text{a } C_k^{d}\text{-tree} \overset{def}{\iff} |\{ i \mid (\text{the } i\text{th comp. of } L) \cap T \neq \phi \}| = k+1$$

distinct

(any disk-leaf grasps different comp. of L)

Thm (H. Miyazawa - Yasuhara '06, Habiro '07)

L : an n-comp. Brunnian link ($\overset{def}{\iff}$ any proper sublink is trivial)

${}^{\exists} T_1 \cup \cdots \cup T_m$ = a disj. union of C_{n-1}^{d}-trees for the n-comp. trivial link O

s.t. $L \overset{a.i}{\sim} O_{\bigcup_{i=1}^{m} T_i}$

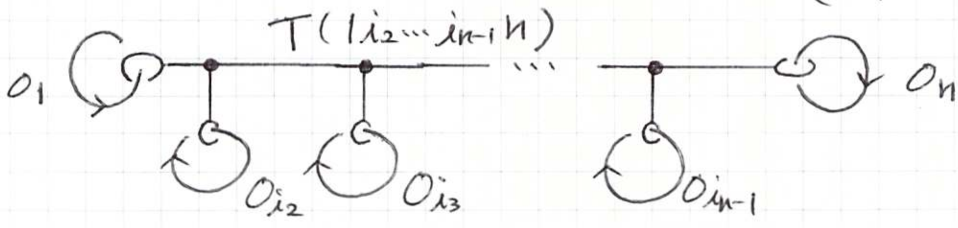
Thm (Milnor '54)

$O = O_1 \cup \cdots \cup O_n$ = the n-comp. trivial link

$T(1\, i_2\, i_3 \cdots i_{n-1}\, n)$: C_{n-1}^{d}-tree for O as illustrated in the following Figure.

$L := O_{T(1 i_2 \cdots i_{n-1} n)}$

$$\bar{\mu}_L(1\, j_2 \cdots j_{n-1}\, n) = \begin{cases} 1 & (j_2 \cdots j_{n-1} = i_2 \cdots i_{n-1}) \\ 0 & (\text{otherwise}) \end{cases}$$



$\overset{a.i}{\sim}$ Milnor link M_n

§. Milnor invariants and claspers

<u>Thm</u> (Habiro '00)

The length $\leq k$ Milnor's $\bar{\mu}$-inv. are inv. of C_k-equiv.

<u>Rmk</u> L, L' : links

$$L \overset{C_k}{\sim} L' \Rightarrow \pi_1(S^3 \setminus L) / (\pi_1(S^3 \setminus L))_{k+1} \cong \pi_1(S^3 \setminus L') / (\pi_1(S^3 \setminus L'))_{k+1}$$

<u>Def.</u> T : a simple C_k-tree for L

T : a C_k^s-tree $\overset{\text{def}}{\Leftrightarrow} |\{ i \mid (\text{the } i\text{th comp. of } L) \cap T \neq \emptyset \}| = 1$

The self C_k-equiv. := an equiv. relation on links generated by C_k^s-trees.

<u>Ex</u>

self C_1-equiv. = link-homotopy

self C_2-equiv. = self Δ-equiv.

<u>Def.</u> $I = i_1 \cdots i_m$: a sequence

$r(I)$: the maximum number of times that any index apperes in I

(ie, $r(1123) = 2$, $r(123/223) = 3$)

<u>Thm</u> (Fleming - Yasuhara '09)

Milnor's $\bar{\mu}$-inv. for I with $r(I) \leq m$ are self C_m-equiv. inv.

link-homotopy classification

	n-comp. link	n-string link
$n=2,3$	Milnor '54 ($\bar{\mu}$-inv.)	Habegger–Lin '90 (μ-inv.)
$n=4$	Levine '88 ($\bar{\mu}$-inv. + α)	
$n\geq 5$	Habegger–Lin '90 (algorithm)	

Thm (Milnor '54)

$L \overset{l.h.}{\sim}$ trivial link $\Leftrightarrow$ $\bar{\mu}_L(I)=0$ for $^\forall I$ with $r(I)=1$.

Thm (Yasuhara '09)

$L \overset{\text{self } C_2}{\sim}$ trivial link $\Leftrightarrow$ $\bar{\mu}_L(I)=0$ for $^\forall I$ with $r(I)\leq 2$.